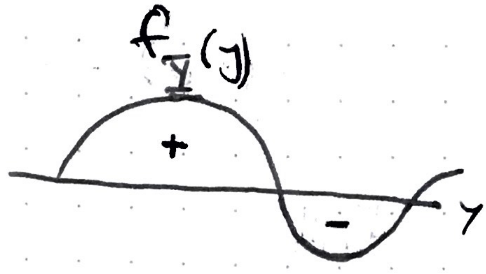
4/30/19 Lecture 9

Rules for pdf



1) $f_Y(y) \geq 0$

2) $\int_{-\infty}^{\infty} f_Y(y) = 1$

Make sure you don't get lost in the calculus portion and use Wolfram Alpha to take care of it.

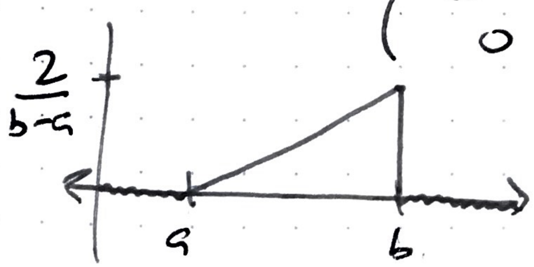
Wolfram Alpha search: integrate c(x-a)/(b-a) for x=a to b

Definite integral:

$$\int_a^b \frac{c(x-a)}{b-a}\,dx = \frac{1}{2}c(b-a)$$

The triangular distribution that starts at 0 for y=a and rises linearly to $\frac{2}{b-a}$ at y=b has density (pdf)

$$f_Y(y) = \begin{cases} \frac{2(y-a)}{(b-a)^2} & \text{for } a \leq y \leq b \\ 0 & \text{else} \end{cases}$$



WA: plot 2(y-1)/(3-1)^2 for y=1 to 3

Will plot the graph for the special case.

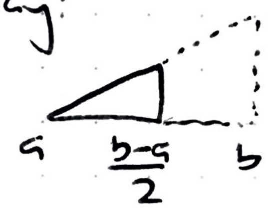
Ex: With the triangular distribution, what's

$P(a \le X \le \frac{b-a}{2})$?

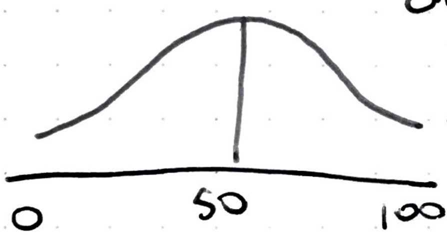
(Y)

Hard Way: $$\int_a^{\frac{b-a}{2}} \frac{2(x-a)}{(b-a)}\,dx = \frac{(3a-b)^2}{4(b-a)^2}$$

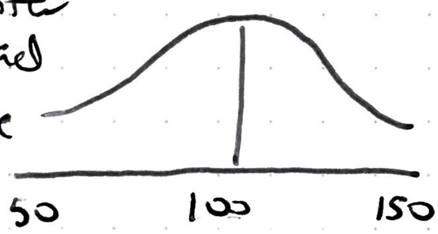
Easier Way:

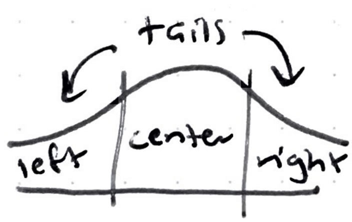


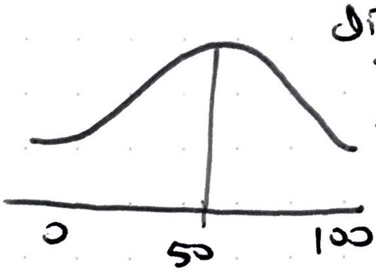
area of a triangle $= \frac{bh}{2}$



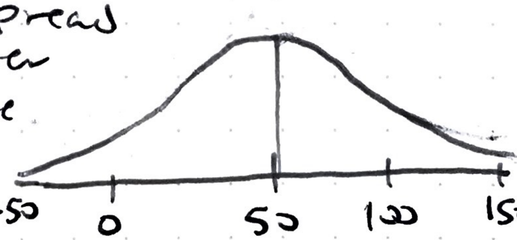
different center
same spread
same shape

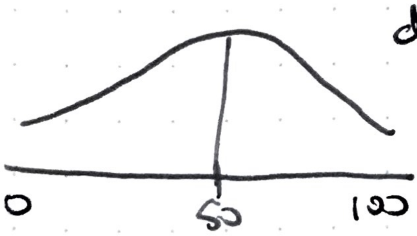




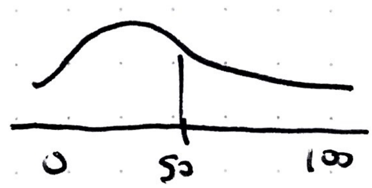


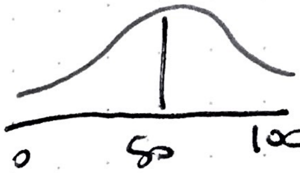
different spread
same center
same shape





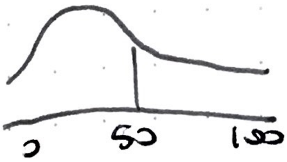
different shape
same center
same spread



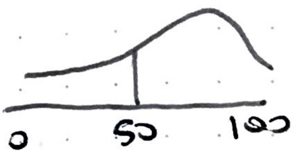


symmetric not skewed (1)

point of symmetry

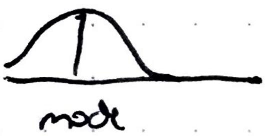


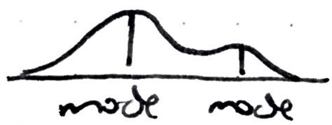
asymetric positive skew
long right-hand tail (2)



asymetric negative skew
long left-hand tail (3)

What's the difference between these distributions?





unimode multimodal (bimodal)

: height of all AMS 131 students
(mixture distribution)

like the bell curve

point of symmetry

(1) height of adult females → symmetric

(2) family income — positive skew

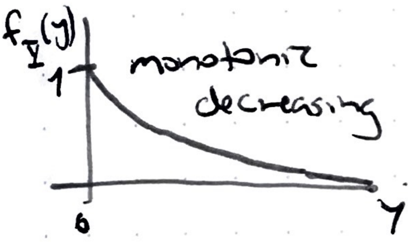
(3) scores on take home test 1 — negative skew

Sometimes it's mathematically convenient to work w/ an unbounded continuous rv, just as was true in the Poisson case study for discrete rvs

Ex: Y = voltage in an electrical system

In practice, Y cannot be infinite, but you may not know ahead of time what its maximum practical value is, so model it as unbounded but without much probability for extremely large values.

As an example, we're given the pdf $f_Y(y) = \frac{1}{(1+y)^2} I(y>0)$



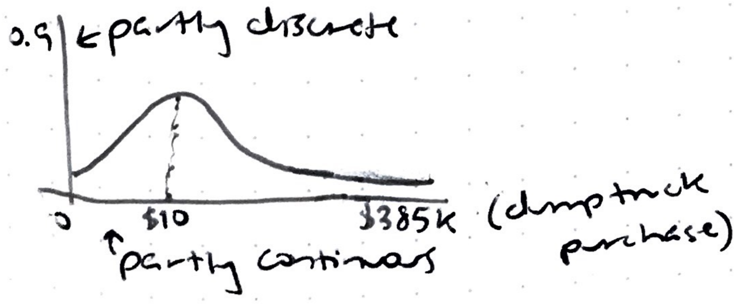
$$f_Y(y) = \begin{cases} \frac{1}{(1+y)^2} & \text{for } y>0 \\ 0 & \text{else} \end{cases}$$

You can check that $\int_{1000}^{\infty} \frac{1}{(1+x)^2}\,dx = \frac{1}{1001} = 0.001$, so the right tail beyond $Y=1000$ has almost no probability, matching the correct qualitative behavior.

Sometimes a rv will be neither discrete nor continuous, people then say that it has a mixed (discrete/continuous) distribution

Ebay

$$Y = \text{GMB (gross merchandise bought) in } (0, \underset{\text{eg. 2 weeks}}{T})$$



Ex:
In medical clinical trials of people with potentially fatal diseases, the outcome variable Y_i for person i in the treatment group might be Y = survival time in days from the beginning of the trial

Some patients may still be alive at the time Tend at which the trial finishes

Your probability model for Y_i would then have a continuous part for $0 \le Y_i \le T_{end}$ and a discrete lump of probability p at $Y_i = T_{end}$ signifying $(Y_i > T_{end})$ but we don't know what Y_i would have been if we could have observed it:

$$\int_0^{T_{end}} f_{Y_i}(y)\,dy = (1-p) \quad \text{and} \quad P(Y_i > T_{end}) = p$$

$(Y_i > T_{end})$: the person was still alive at the end of the trial so we don't know what their survival time was

All we know is they lived at least that long

This is an example of not getting full data on the subjects of interest to you, called sensoring

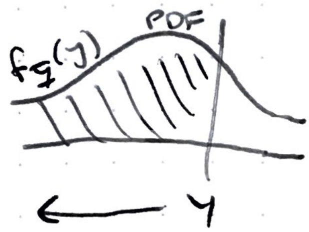
This one is right-censoring

$\boxed{\text{Unifying idea connecting discrete \& continuous rvs}}$: Discrete $\leftrightarrow$ pf (pmf)
Continuous $\leftrightarrow$ pdf
Mixed $\leftrightarrow$ (pf + pdf)

Q: Is there something that uniquely characterizes the distribution of Y, both when Y is discrete & when it's continuous & when it's mixed?

A: Yes, the cumulative distribution function (cdf) $F_Y(y)$

<u>Def</u>: The cumulative distribution function (CDF) of a rv Y is defined to be $F_Y(y) = P(Y \leq y)$ for all $-\infty < y < \infty$



CDF cont.

$$F_Y(y) = P(Y \leq y) = \int_{-\infty}^{y} f_Y(t)\, dt$$

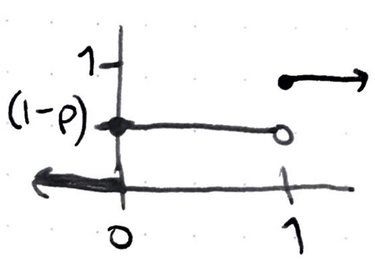
dummy variable of integration

$$0 \leq F_Y(y) \leq 1$$

<u>Ex</u>: $Y \sim \text{Bernoulli}(p)$

PMF
$$P(Y = y) = \begin{cases} p & \text{for } y = 1 \\ 1-p & 0 \\ 0 & \text{else} \end{cases}$$

Clever way to write this: $P(Y=y) = p^y (1-p)^{1-y} I_{\{0,1\}}(y)$



$F_Y(y) = P(Y \leq y)$

The cdf of Y is 0 for $y < 0$; at $Y = 0$ it jumps up to $(1-p)$ and stays there for $0 \leq y < 1$; and at $Y = 1$ it jumps up to 1 & stays there for $y \geq 1$

CDF can never dip down $F_Y(y)$ has to be <u>non-decreasing</u>

If $y_1 < y_2$ then $F_Y(y_1) \le F_Y(y_2)$

$\lim_{y \to -\infty} F_Y(y) = 0$ and $\lim_{y \to \infty} F_Y(y) = 1$

CDFs F_Y can be all of $\mathbb{R}$ (when Y is continuous) but certainly don't have to be (see the CDF of the Bernoulli (p) distribution)

Technical fact:

Def: $F_Y(y^-) = \lim_{\substack{y^* \to y \\ y^* < y}} F_Y(y^*) = \lim_{y^* \uparrow y} F_Y(y^*)$

limit from the left

(y goes to y from below)

<u>Def:</u> $F_Y(y) = F_Y(y^+)$ for all $-\infty < y < \infty$

People call this continuity from the right or continuity from above

If $F_Y(y^-) = F_Y(y^+) = F_Y(y)$ then F_Y is continuous at y

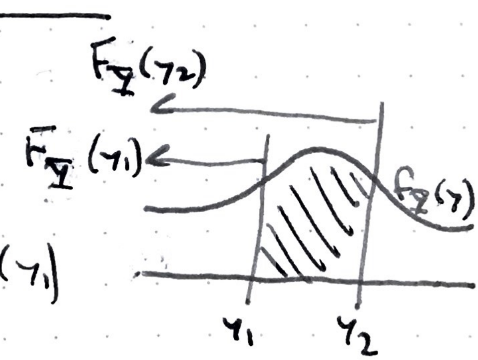
<u>Consequences of the CDF definition</u>

1) $P(Y > y) = 1 - F_Y(y)$

2) For all y_1, y_2 with $y_1 < y_2$

$P(y_1 \le Y \le y_2) = F_Y(y_2) - F_Y(y_1)$

↓ Cont.



If Y is continuous, there's an infinite connection between $F_Y(y)$ and $f_Y(y)$
(CDF) (PDF)

Y continuous: $y_1 < y_2$

$$P(y_1 < Y < y_2) = F_Y(y_2) - F_Y(y_1)$$
$$= \int_{y_1}^{y_2} f_Y(y)\, dy$$

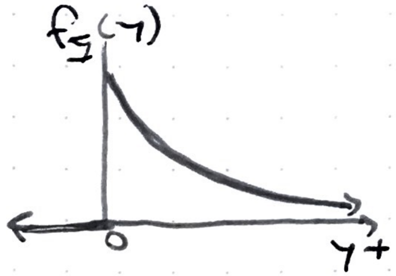
<u>Theorem</u>: If Y is a continuous rv, with pdf $f_Y(y)$ and CDF $F_Y(y)$ then

$F_Y(y) = \int_{-\infty}^{y} f_Y(t)\, dt$ and at all continuity points of f $\frac{d}{dy} F_Y(y) = f_Y(y)$

Y continuous $\leftrightarrow$ the derivative of $F_Y(y)$ is $\overset{\text{PDF}}{f_Y(y)}$ and $F_Y(y)$ is an anti-derivative (integral) of $f_Y(y)$

<u>Def</u>: Y follows an exponential distribution $\mathcal{E}(\lambda)$ with (rate) parameter $\lambda > 0$

$$\underset{\text{pdf}}{f_Y(y)} = \begin{cases} \lambda e^{-\lambda y} & y > 0 \\ 0 & y \le 0 \end{cases}$$



$$f_Y(y) = \begin{cases} ce^{-\lambda y} & \text{for } y \ge 0 \\ 0 & \text{else} \end{cases}$$

$(\lambda > 0)$

The exponential dist. has a fundamental connection to the Poisson distribution in Poisson processes that we'll explore later

CDF of Y : $F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ 1-e^{-\lambda y} & \text{for } y \geq 0 \\ 1 & \text{as } y \uparrow \infty \end{cases}$

for $y \geq 0$

$$F_Y(y) = \int_0^y \lambda e^{-\lambda t} \, dt$$

$$= 1 - e^{-\lambda t}$$