6/4/19

Tuesday, June 4, 2019 1:29 PM

This time: Large random samples
Next time: Markov chains

IID random sample $X_1, \dots, X_n$
$\mu = E(X_i)$
sample mean: $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

It would be nice if $\bar{X}_n$ approached the right answer μ as n increases; how to quantify that idea?

Markov inequality

$$P(X \geq t) \leq \frac{E(X)}{t}$$

ex: $E(X) = 1$, X non-negative $\rightarrow$ $P(X \geq 100) \leq \frac{1}{100}$

Chebyshev Inequality

$$P[\,|X - E(X)| \geq t\,] \leq \frac{V(X)}{t^2}$$

$\Rightarrow$ BACK to $\bar{X}_n$

$E(X_i) = \mu \qquad V(X_i) = \sigma^2 < \infty$

$E(\bar{X}_n) = \mu \qquad V(\bar{X}_n) = \frac{\sigma^2}{n}$

$$P(|\bar{X}_n - \mu| \geq t) \leq \frac{\sigma^2}{n t^2} \quad \text{for all } t > 0$$

rewritten as:

$$P(|\bar{X}_n - \mu| < t) \geq 1 - \frac{\sigma^2}{n t^2}$$

Def: A sequence $Z_1, Z_2, \dots$ of r.v. is said to converge in probability to constant b if for all $\epsilon > 0$, $\lim_{n \to \infty} P(|Z_n - b| < \epsilon) = 1$;
this is denoted $Z_n \xrightarrow{P} b$.

(weak)
Law of
Larg Numbers

$X_i \sim$ IID a dist. with mean

Corollary: If $Z_n \xrightarrow{P} b$ and $g(z)$ is continuous at $z = b$ then $g(Z_n) \xrightarrow{P} g(b)$

Central Limit Theorem

$X_i \sim$ IID any dist. with mean μ and finite variance $0 < \sigma^2 < \infty$,
$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \rightarrow$ for large n $\quad \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$\boxed{\text{Careful Statement}}$ Def: $X_1, X_2, \dots$ a sequence of r.v., let F_n be the CDF of X_n
$\rightarrow$ if there exists a CDF F^* such that $\lim_{n \to \infty} F_n(x) = F^*(x)$ for all x at which $F^*(x)$ is continuous, then people say

X has mean μ_X
expand $g(\bar{X}_n)$ around μ_X:

$$g(\bar{X}_n) \doteq g(\mu_X) + g'(\mu_X)(\bar{X}_n - \mu_X)$$

$$E(g(\bar{X}_n)) = E[\,g(\mu_X) + g'(\mu_X)(\bar{X}_n - \mu_X)\,]$$

$$= g(\mu_X) + g'(\mu_X)\,[\underbrace{E(\bar{X}_n) - \mu_X}_{0}]$$

$$E[g(\bar{X}_n)] = g(\mu_X)$$

$$V[g(\bar{X}_n)] = [g'(\mu_X)]^2 \frac{\sigma_X^2}{n}$$

Δ METHOD

$$E(W) \doteq g(\mu_V) \text{ and}$$

$$V(W) \doteq [g'(\mu_V)]^2 \sigma^2_V$$

Continuity correction: