(23)

$$P_K\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P_K(A_i) \quad (*)$$

disjoint

(this countable additivity turns out to be absolutely necessary but is hard to motivate: it's a small piece of genius on Kolmogorov's part that he assumed this not just for a finite number of disjoint events) — and

if $A_1, \ldots, A_n$ are disjoint then

$$P_K\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P_K(A_i) \text{ follows from } (*)$$

— but also for a countable collection.

(9 Apr 19)

Consequences that follow from Kolmogorov's Axioms

(From now on I'll drop the subscript K.)
(Kolmogorov)

① $P(\phi) = 0$

DS: Pr

p

(2) $P(A^c) = 1 - P(A)$.

(3) If $A \subset B$ then $P(A) \le P(B)$.

(4) For all events A,

$0 \le P(A) \le 1$ (the easy rule)

(5) For all events A, B,

general addition rule for $\boxed{\text{or}}$

$$P(\underset{\text{or}}{A \cup B}) = P(A) + P(B) - P(\underset{\text{and}}{A \cap B}).$$

(6) (attributed to the Italian mathematician Carlo Bonferroni (1892 - 1960)): For any events $A_1, A_2, \dots, A_n$,

$$P\left(\bigcup_{i=1}^{n} A_i \right) \le \sum_{i=1}^{n} P(A_i) \quad \text{and}$$

$$\boxed{P\left(\bigcap_{i=1}^{n} A_i \right) \ge 1 - \sum_{i=1}^{n} P(A_i^c)}.$$

useful in statistics

Tay-Sachs disease in more detail

NNNNN	0
TNNNN	1
NTNNN	
NNTNN	
NNNTN	
NNNNT	
TTNNN	2
TNTNN	
TNNTN	
TNNNT	
NTTNN	
NTNTN	
NTNNT	
NNTTN	
NNTNT	
NNNTT	
⋮	⋮
TTTTT	5

$\boxed{\text{\# of T-S babies} = Y}$ Let's see if we can work out

$P(Y=1)$, $P(Y=2)$, $\ldots$, $P(Y=5)$; we already worked out

$$P(Y=0) = P(\text{exactly } 0 \text{ T-S babies})$$

$$= P\begin{pmatrix}\text{1st} \\ \text{baby} \\ \text{not} \\ \text{T-S}\end{pmatrix} \,\&\, \begin{matrix}\text{2nd} \\ \text{baby} \\ \text{not} \\ \text{T-S}\end{matrix} \,\&\, \ldots \,\&\, \begin{matrix}\text{5th} \\ \text{baby} \\ \text{not} \\ \text{T-S}\end{matrix}\Big)$$

independence

$$= P\begin{pmatrix}\text{1st} \\ \text{baby} \\ \text{not} \\ \text{T-S}\end{pmatrix} \cdot P\begin{pmatrix}\text{2nd} \\ \text{not} \\ \text{T-S}\end{pmatrix} \cdot \ldots \cdot P\begin{pmatrix}\text{5th} \\ \text{not} \\ \text{T-S}\end{pmatrix}$$

identical distribution

$$= \left[1 - P\begin{pmatrix}\text{1st} \\ \text{baby} \\ \text{T-S}\end{pmatrix}\right] \cdot \ldots \left[1 - P\begin{pmatrix}\text{5th} \\ \text{baby} \\ \text{T-S}\end{pmatrix}\right] = (1-p)^5 = 1\, p^0 (1-p)^5 \doteq 24\% \text{ with } p = \frac{1}{4}$$

A similar line of reasoning gives

$P(Y=5) = P(TTTTT) = p^5 = 1\,p^5(1-p)^0$

$\boxed{\text{what about } P(Y=1)?}$ The table on the previous page lists all of the outcomes with 1 T-s baby: they all have 1 T and 4 Ns, so each one has probability $p(1-p)^4$, and there are 5 of them, so $P(Y=1) = 5\,p^1(1-p)^4$.

By similar reasoning $P(Y=2) = 10\,p^2(1-p)^3$

The outcomes with $(Y=3)$ are mirror images of those with $(Y=2)$: $\left\{\begin{matrix} TTNNN \\ NNTTT \end{matrix}\right\}$

so there must also be 10 elements of S with $(Y=3)$ and $P(Y=3) = 10\,p^3(1-p)^2$.

And finally, $(Y=4)$ is a mirror image of $(Y=1)$ so $P(Y=4) = 5\,p^4(1-p)^1$

# of T-s babies y	$P(Y=y)$	with $p=\frac{1}{4}$
0	$1\,p^0(1-p)^5$	0.2373
1	$5\,p^1(1-p)^4$	0.3955
2	$10\,p^2(1-p)^3$	0.2637
3	$10\,p^3(1-p)^2$	0.0879
4	$5\,p^4(1-p)^1$	0.0146
5	$1\,p^5(1-p)^0$	0.0010
	1	1.0000

Soon we'll call $\boxed{\text{upper case}}$ Y a <u>random variable</u> (symbolizing the data generating process) and $\boxed{\text{lower case}}$ use y to stand for a possible value of Y.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1
\end{array}
$$

So it looks like

$n =$ # children

$$P(Y = y) = \boxed{?}\, p^y (1-p)^{5-y}.$$

we could even be a bit more symbolic and note that $n = 5$ is the number of times the basic dichotomy (T vs. N) occurs in this case study, so $P(Y = y) = \boxed{?}\, p^y (1-p)^{n-y}$.

What about $\boxed{?}$

You can see that the multipliers $\boxed{?}$ come from Pascal's Triangle, but can we write down a formula for them?

Permutations & Combinations

Ex. You have an ordinary deck of $n = 52$ playing cards.

How many possible poker hands of 29
$k=5$ cards can you draw at random
without replacement from the deck?

(8 of diamonds)

It's like filling in 5 slots: $\underset{\blacklozenge}{8}$ _ _ _ _

the first slot can be filled in $n=52$
ways, and the second in $(n-1)=51$
ways, ..., the 5^{th} slot in $(n-k+1)=48$
ways; so the total # of ways you
can do this is $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$

$= n(n-1)\cdots(n-k+1) = 311{,}875{,}200$

ways. This is called the number
of <u>permutations</u> of 52 things taken
5 at a time.

Definition | The number of permutations of n distinct things taken k at a time is written $P_{n,k} = n(n-1)\cdots(n-k+1)$.

capital

How many possible orderings of a 52-card deck are there? Now there are 52 slots, e.g., $\underset{\text{of}\,\diamondsuit}{J}$ $\underset{\text{of}\,\heartsuit}{3}$ $\cdots$ $\underset{\text{of}\,\clubsuit}{A}$, so the total

$52 \cdot 51 \cdots\cdots 1$

number must be $52 \cdot 51 \cdots\cdots 1 =$ Def.

$$\boxed{n(n-1)\cdots \underline{1} \overset{\Delta}{=} n! \text{ read } n \text{ factorial}}$$

$= 80658175170943878571660636856403766975289$
$505440883277824000000000000 \doteq 8.1 \cdot 10^{67}$

free wolfram alpha

(maple) (Apr 19)