What's the place \( Y_p \) on the positive part of \( R \) where \( P(0 \leq Y \leq Y_p) = p \)?

Well, \( \mathbb{P}(0 \leq Y \leq Y_p) = \mathbb{P}(\frac{-\ln(Y_p)}{\alpha} \leq \frac{-\ln(1-p)}{\alpha}) = \mathbb{P}(\frac{-\ln(Y_p)}{\alpha}) = \mathbb{P}(\frac{-\ln(1-p)}{\alpha}) = p \).

So, \( Y_p = F^{-1}(p) \).

\( Y_p \) is called the \( \frac{p}{100} \) quantile or the \( 100 \frac{p}{100} \)th percentile of the distribution of \( Y \).
Some care is required when \( X \) is discrete or mixed.

For all \( 0 < p < 1 \) define

\[
F^{-1}_{\Sigma}(p) = \text{the smallest } y \text{ value such that } F_{\Sigma}(y) \geq p
\]

Then \( F^{-1}_{\Sigma}(p) \) is the \( p \)-th quantile of \( \Sigma \) and \( F^{-1}_{\Sigma} \) is the quantile function.

One way to define the center of a distribution is to find the 50th percentile.
Definition: The $\frac{1}{2}$ quantile is the 50% percentile of a distribution, called the median of the dist.

Measure of spread for the distribution of a rv $X$.

One way to define the spread of a dist. is to see how far apart its 75th and 25th percentiles are.

Definition: The $\frac{1}{4}$ quantile is the 25th percentile, the lower quartile $Q_1$.

The $\frac{3}{4}$ quantile is the 75th percentile, the upper quartile $Q_3$.

$Q_1 = F^{-1}(0.25)$, $Q_3 = F^{-1}(0.75)$, and $[F^{-1}(0.75) - F^{-1}(0.25)] = \text{interquartile range (IQR)}$. 

$F^{-1}$ is the inverse CDF.
Example \( X \sim \text{Uniform}(a,b) \); then \( F_X(x) \):

\[
F_X(x) = \begin{cases} 
0 & \text{for } x < a \\
\frac{x-a}{b-a} & a \leq x \leq b \\
1 & \text{for } x > b
\end{cases}
\]

Easy to invert \( F_X \):

\[
F_X^{-1}(p) = (1-p)a + pb \quad \text{for } 0 < p < 1
\]

And (no surprise) the median is \( \frac{a+b}{2} \).

Studying two random variables at a time:

Def. \( X, Y \) rvs: the joint (or bivariate) distribution of \( (X,Y) \) is the collection \( \mathbb{P}(X,Y) \) of all probabilities for all sets \( C \in \mathbb{R}^2 \) such that \( (X,Y) \in C \) isn't weird.
Case 1) \((X \text{ and } Y \text{ both discrete})\)

Def: \(X, Y \text{ rv.} \Rightarrow \) if there are only finitely or countably infinitely many possible values \((x, y)\) for \((X, Y)\), \(X\) and \(Y\) have a discrete joint dist.

Def: The joint probability function (joint pf) of \((X, Y)\) discrete is the function \(f_{XY}(x, y) = P(X = x, Y = y)\).

The set \(\{(x, y) : f_{XY}(x, y) > 0\}\) is the support of \(f_{XY}\).

Consequences:

1. \(\sum_{x, y} f(x, y) = 1\) \(\forall (x, y) \in \mathbb{R}^2\) (unit area)

2. For any set \(C\) of ordered pairs \((x, y)\), \(P[(X, Y) \in C] = \sum_{(x, y) \in C} f_{XY}(x, y)\) (non-null)
Def. Two rv $X$ and $Y$ have a continuous joint distribution if you can find a nonnegative function $f_{X,Y}(x,y)$ defined for all $(x,y) \in \mathbb{R}^2$ (the real plane) such that for every (non-weird) subset $C$ of the plane $P(\{(X, Y) \in C\}) = \int_{C} f_{X,Y}(x,y) \, dx \, dy$. $f_{X,Y}(x,y)$ is the joint pdf of $(X, Y)$.

The set $\{(x,y): f_{X,Y}(x,y) > 0\}$ is the support of the dist. of $(X, Y)$.

Immediate Consequences:
1. For all $(x,y)$ in $\mathbb{R}^2$, $f_{X,Y}(x,y) \geq 0$, and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \, dy = 1$. 

Causal: $X, Y$ continuous
For all continuous pdfs \( f_{X,Y}(x,y) \),

1. Every individual point, and every countably infinite sequence of points, has probability 0 under \( f_{X,Y} \).

2. If \( (X,Y) \) have a continuous joint distribution, \( X \) and \( Y \) each have a continuous (marginal) univariate distribution when considered separately.

3. If \( g \) is a continuous function of one real variable defined on \((a,b)\), then the sets \( \{(x,y) : y = g(x), a < x < b\} \) and \( \{(x,y) : x = g(y), a < y < b\} \) also have probability 0.
This means that the converse of (2) is (unfortunately) not true: If \( X \) has a continuous distribution on \( \mathbb{R} = \mathbb{R}^1 \) and \( Z = X \), then both \( X \) and \( Z \) have continuous distributions but \( P \left( (X, Z) \text{ lies on the line } y = x \right) \neq 0 \). So \( (X, Z) \) can't have a continuous joint distribution on \( \mathbb{R}^2 \).

**Example**

Suppose that joint distribution \( (X, Z) \) have joint pdf can lead to tricky integrals.

\[
f_{XY}(x, y) = \begin{cases} 
    c x^2 y & \text{for } 0 \leq x^2 y \leq 1 \\
    0 & \text{else}
\end{cases}
\]

Let's work out the normalizing constant.

The support of \( f_{XY} \) is the shaded region here.
\begin{align*}
1 &= \int_0^\infty \int_0^\infty F_E(x, y) \, dx \, dy \\
&= \int_0^1 \int_0^1 f(x, y) \, dy \, dx \\
&= \int_1^{x_1} \int_{x_2}^{x_1} c \, x \, y \, dy \, dx \\
&= \int_1^1 \int_0^{\sqrt{\frac{2}{x}}} c \, x^2 \, (\frac{2}{x} \, 1) \, dx \\
&= \int_1^1 \int_0^{\sqrt{\frac{4}{x^2}}} c \, x^2 \, \left( \frac{1}{2} - \frac{x^2}{2} \right) \, dx \\
&= \frac{1}{2} c \int_0^1 \int_0^1 x^2 \, 2x \, dx - \frac{1}{2} c \int_0^1 \int_0^1 x^2 \, 4x^2 \, dx \\
&= \frac{1}{2} c \left( \frac{x^3}{3} \bigg|_1^1 \right) - \frac{1}{2} c \left( \frac{x^5}{5} \bigg|_1^1 \right) = \frac{4}{21} c = 1
\end{align*}
So \( c = \frac{21}{4} \)

The other way to parameterize the support \( S \) is to let \( y \) go from 0 to 1 while \( x \) goes from \(-\sqrt{y}\) to \(\sqrt{y}\).

\[
1 = \iint_{S} f_{X\mid Y}(x, y) \, dx \, dy
\]

\[
= \int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} c x^2 y \, dx \, dy
\]

\[
= \int_{0}^{1} c y \left( \int_{-\sqrt{y}}^{\sqrt{y}} x^2 \, dx \right) \, dy
\]

\[
= \int_{0}^{1} c y \left( \frac{x^3}{3} \right) \bigg|_{-\sqrt{y}}^{\sqrt{y}} \, dy
\]

\[
= c \int_{0}^{1} y \cdot \frac{1}{3} \left( y^{3/2} - (-y)^{3/2} \right) \, dy
\]
\[ z = \frac{2}{3} \int_0^1 2 \gamma^{5/2} \, d\gamma = \frac{2}{3} \left( \frac{\gamma^{3/2}}{3/2} \right) \bigg|_0^1 = \frac{4}{21} \]

as before (\(\iiint dx \, dy \, dz = \iiint dz \, dy \, dx\))

always have to agree, of course.

\(\text{Example, continued}\)

\[ P(X \geq 2) \]

\[ (x, y) \]

\[ x = 1 \]

\[ (0, 0) \]

\[ (1, 1) \]

\[ y = x \]

\[ y = x^2 \]

\[ \int_{S_0} f_{X,Y}(x, y) \, dy \, dx \]

\[ = \int_0^1 \int_0^x \frac{21}{4} x^2 \, dy \, dx = \frac{3}{20} \]

\(\text{let's compute}\)

\[ \text{the relevant part of } S \text{ where } x \geq y \text{ is sketched here, so}\]

\[ \int_{S_0} \frac{21}{4} x^2 \, dy \, dx \]

\(\text{then integrate result from } x = 1 \)}
You can have bivariate distributions in which one of \((X,Y)\) is discrete and the other is continuous. \(Y\) rv such that \(X\) is discrete and \(Y\) is continuous → suppose you can find a function \(f_{XY}(x,y)\) defined on \(\mathbb{R}^2\) such that for every pair of (non-void) subsets \(A\) and \(B\) of \(\mathbb{R}\) (assuming exists)
\[
P(X \in A \text{ and } Y \in B) = \int_B \int_A f_{XY}(x,y) \, dx \, dy.
\]
Then \(f\) is the joint pmf/pdf of \((X,Y)\)

**Immediate Consequence**

If \(X\) takes on values \(x_1, x_2, \ldots,\)
then
\[
\sum_{-\infty}^{\infty} f(x_i, y) \, dy = 1.
\]
Example: Randomized controlled (clinical) trial; patients in \( \Theta \) get a treatment, patients in \( \Theta \) get a placebo. Outcome is success (e.g., cancer goes into remission) or failure; let \( X_i = \begin{cases} 1 \text{ if patient } i \text{ is a success} \\ 0 \text{ else} \end{cases} \) and let \( \Theta \) be the proportion of patients in the population of all patients who might get the treatment who would have no relapse if they had been in the study. Then, our uncertainty about \( \Theta \) is continuous on \((0, 1)\) and \((X_i, \Theta)\) has a bivariate distribution.
If you model \( (X | \theta) \) as Bernoulli(\( \theta \)) and \( \theta \sim \text{uniform}(0,1) \), the joint pdf of \((X, \theta)\) would be

\[
f_{X, \theta}(x, \theta) = \begin{cases} \theta (1-\theta) & \text{for } (x = 0, 1) \\ 0 & \text{else} \end{cases}
\]

Then (e.g.) \( P(X = 1) = P(X = 1 \text{ and } \theta \text{ is anything between 0 and 1}) \)

\[
= \int_0^1 \theta (1-\theta) \, d\theta = \int_0^1 \theta \, d\theta = \frac{1}{2}.
\]

(Bivariate CDFs)

The joint CDF of two rvs \( X \) and \( Y \) is the function \( F_{X,Y}(x,y) \) satisfying \( F_{X,Y}(x,y) = \Pr(X \leq x \text{ and } Y \leq y) \) for all \(-\infty < x < \infty\) and \(-\infty < y < \infty\).