Thus the triangular distribution that starts at $y=a$ and rises linearly to

$$f(y) = \begin{cases} \frac{2}{b-a} & \text{for } y=1 \\ \frac{2(y-a)}{(b-a)^2} & \text{for } y \text{ from } a \text{ to } b \\ 0 & \text{else} \end{cases}$$

You can see that calculating probabilities with continuous rvs requires you to dust off your integral calculus.

Example: with the triangular distribution (\text{I}) above, what's $P(a \leq X \leq \frac{b-a}{2})$?

Hard(?): \[ \int_{a}^{b} \frac{2(x-a)}{(b-a)^2} \, dx = \frac{(b-a)^2}{4(b-a)^2} \]

Easy(\text{I}): \[
\begin{align*}
\text{area of triangle} &= \frac{1}{2} \times \text{base} \times \text{height} \\
&= \frac{1}{2} \times \frac{b-a}{2} \times \frac{b-a}{2} \\
&= \frac{(b-a)^2}{4}
\end{align*}
\]
Sometimes it's mathematically convenient to work with unbounded continuous r.v.'s, just as was true in the Poisson case study for discrete r.v.'s.

Example (DS p. 185)

\[ \mathbb{I} = \text{voltage in an electrical system} \]

In practice \( \mathbb{I} \) cannot be infinite, but you may not know ahead of time what its maximum practical value is, so model it as unbounded but without much probability for extremely large values. If give us an example the pdf

\[ f_{\mathbb{I}}(y) = \frac{1}{(1+y)^2} \mathbb{I}(y > 0) \]

\[ = \frac{1}{(1+y)^2} \left. \right|_0^\infty = -1 \left( 0 - 1 \right) = 1 \checkmark \]

\[ \int_0^\infty \frac{1}{(1+y)^2} \, dy = 1 \]
You can check that \( \int_{1000}^{\infty} \frac{1}{1 + x} \, dx = 1 \), so the right tail beyond \( x = 1000 \) has almost no probability, matching the correct qualitative behavior.

As a rv will be neither discrete nor continuous, people then say that it has a mixed (discrete/continuous) distribution. Example:

In medical clinical trials of people with potentially fatal disease, the outcome variable \( Y_i \) for person \( i \) in (say) the treatment group might be
$T_i$ = survival time in days from the beginning of the trial; however, and a good thing too, some patients may still be alive at the time $T_{end}$ at which the trial finishes. Your model for $T_i$ would then have a continuous part for $0 \leq T_i \leq T_{end}$ and a discrete lump of probability $p$ at $T_i = T_{end}$ signifying $(T_i > T_{end})$ but we don't know what $T_i$ would have been if we could have observed it: (right-censoring)

\[
\int_0^{T_{end}} f_{T_i}(y) dy = (1-p) \quad \text{and} \quad P(T_i > T_{end}) = p.
\]

(Rebut example) (see p. 10 of doc. can wts)
Unifying idea connecting discrete & continuous rvs

\[ \text{Discrete} \leftrightarrow \text{pf (pmf)} \]

\[ \text{Continuous} \leftrightarrow \text{pdf} \]

\[ \text{Mixed} \leftrightarrow (\text{pf} + \text{pdf}) \]

Is there something that uniquely characterizes the distribution of \( X \), both when \( X \) is discrete & when it's continuous & when it's mixed?

**A:** Yes, the cumulative distribution function (cdf)

\[ F_X(y) \]

**Definition:**

The cumulative distribution function (cdf) of a rv \( X \) is defined to be

\[ F_X(y) = P(X \leq y) \text{ for all } -\infty < y < \infty \]
Example: $I \sim \text{Bernoulli}(p)$

$$P(I = \gamma) = \begin{cases} p & \text{for } \gamma = 1 \\ 1-p & \text{else} \\ 0 & \text{otherwise} \end{cases}$$

Write this as:

$$P(I = \gamma) = p \gamma (1-p)^{1-\gamma} \mathbb{I}_{\{0,1\}}(\gamma)$$

The cdf of $I$ is 0 for $\gamma < 0$; at $I = 0$ it jumps up to $(1-p)$ and stays there for $0 \leq \gamma < 1$; and at $I = 1$ it jumps up to 1 and stays there for $\gamma \geq 1$.

You can see that in general:

$$0 \leq F_I(\gamma) \leq 1$$
$\lim_{y \to \infty} F(y) = \lim_{y \to \infty} \int_{x=0}^{y} f(x) \, dx$

Fact:

Relevant:

$\lim_{y \to \infty} F(y) = \lim_{y \to \infty} \int_{x=0}^{y} f(x) \, dx$

Furthermore, $\lim_{y \to \infty} F(y) = 0$.

$h(y) = \frac{1}{y}$

Show that $h(y) \to 0$ as $y \to \infty$.
Def. \( F_{\Xi}(y^+) = \lim_{y^* \to y} F_{\Xi}(y^*) \equiv \lim_{y^* \to y} F_{\Xi}(y) \) from above

\( y^* > y \) (\( y^* \) goes to \( y \) from above)

**Technical fact:** \( F_{\Xi}(y) = F_{\Xi}(y^+) \) for all \(-\infty < y < \infty\)

People call this **continuity from the right** or **continuity from above**

Consequence of the CDF definition

1. \( P(\Xi > y) = 1 - F_{\Xi}(y) \)

2. For all \( y_1, y_2 \) \( y_1 < y_2 \)

\( P(y_1 < \Xi \leq y_2) = F_{\Xi}(y_2) - F_{\Xi}(y_1) \)

If

\( F_{\Xi}(y^-) = F_{\Xi}(y^+) = F_{\Xi}(y) \)

then \( F_{\Xi} \) is continuous
Consequence 2) means that if $X$ is continuous, there's an intimate connection between $F_X(y)$ and $f_X(y)$:

$$F_X(y) = \begin{cases} \int_0^y f_X(t) \, dt & \text{for } y < \infty \\ \text{for } y > \infty \end{cases}$$

and thus

**Theorem**

If $X$ is a continuous r.v. with pdf $f_X(y)$ and CDF $F_X(y)$ then

$$F_X(y) = \begin{cases} \int_0^y f_X(t) \, dt & \text{for } y < \infty \\ \text{for } y > \infty \end{cases}$$

and

$$\frac{d}{dy} F_X(y) = f_X(y).$$
In other words, the derivative of \( F_Z(y) \) is \( f_Z(y) \) (and \( F_Z(y) \) is an anti-derivative of \( f_Z(y) \)).

**Definition**

\( Y \) follows an exponential distribution with parameter \( \lambda > 0 \) if

\[
  f_Z(y) = \begin{cases} 
    \lambda e^{-\lambda y} & y > 0 \\
    0 & y \leq 0
  \end{cases}
\]

The exponential distribution has a fundamental connection to the Poisson distribution in Poisson processes that we'll explore later.
It's easy to calculate the CDF of an exponential distribution:

\[ F_\gamma(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ 1 - e^{-\lambda y} & \text{for } y > 0 \end{cases} \]

\( \lambda > 0 \)

\( \gamma \sim \text{Exponential}(\lambda) \)

\[ F_\gamma(y) = \int_{-\infty}^{y} f_\gamma(t) \, dt = \int_{0}^{y} \lambda e^{-\lambda t} \, dt = \left[ -e^{-\lambda t} \right]_{0}^{y} = 1 - e^{-\lambda y} \]

\( \gamma \geq 0 \)