Thus the triangular distribution that starts at 0 for $y=a$ and rises linearly to $\frac{2}{b-a}$ at $y=b$ has density (pdf)

$$f_Y(y) = \begin{cases} \frac{2(y-a)}{(b-a)^2} & a \le y \le b \\ 0 & \text{else} \end{cases} \quad (*)$$

(wd) integrate $2(y-a)/(b-a)^2$ for y from a to b

You can see that calculating probabilities with continuous rvs requires you to dust off your integral calculus.

Example with the triangular distribution (Y) $(*)$ above, what's $P(a \le Y \le \frac{b-a}{2})$?

Hard(?) way: $\displaystyle\int_a^{\frac{b-a}{2}} \frac{2(x-a)}{(b-a)^2}\,dx = \frac{(3a-b)^2}{4(b-a)^2}$

Easy(?) way:

area of triangle = ...

Sometimes it's mathematically convenient to work with unbounded continuous rvs, just as was true in the Poisson case study for discrete rvs.

Example (DS p. 185)

Y = voltage in an electrical system: in practice Y cannot be infinite, but you may not know ahead of time what its maximum practical value is, so model it as unbounded but without much probability for extremely large values. DS give as an example the pdf

$$f_Y(y) = \frac{1}{(1+y)^2} I(y > 0).$$

check: $\int_0^\infty \frac{1}{(1+y)^2} dy = 1$

$$= \frac{(1+y)^{-1}}{-1} \Big|_0^\infty = -1(0-1) = 1 \checkmark$$

You can check that $\int_{1000}^{\infty} \frac{1}{(1+x)^2} dx = \frac{1}{1001} \doteq .001$, so the right tail beyond $Y = 1000$ has almost no probability, matching the correct qualitative behavior. Sometimes a rv will be neither discrete nor continuous; people then say that it has a mixed (discrete/continuous) distribution $\boxed{\text{Definition.}}$ Example: In medical clinical trials of people with potentially fatal diseases, the outcome variable Y_i for person i in (say) the treatment group might be

Y_i = survival time in days from the beginning of the trial; however, and a good thing too, some patients may still be alive at the time T_{end} at which the trial finishes. Your probability model for Y_i would then have a continuous part for $0 \le Y_i \le T_{end}$ and a discrete lump of probability p at $Y_i = T_{end}$ signifying $(Y_i > T_{end})$ but we don't know what Y_i would have been if we could have observed it (right-censoring):

$$\int_0^{T_{end}} f_{Y_i}(y)\, dy = (1-p) \quad \text{and} \quad P(Y_i > T_{end}) = p.$$

(EBay example) (see p. 10 of doc. cam notes)

Unifying idea connecting discrete & continuous rvs

Discrete $\leftrightarrow$ pf (pmf)

Continuous $\leftrightarrow$ pdf

Mixed $\leftrightarrow$ (pf + pdf)

Q: Is there something that uniquely characterizes the distribution of Y, both when Y is discrete & when it's continuous & when it's mixed?

A: Yes, the cumulative distribution function (cdf) $F_Y(y)$

Definition:

The cumulative distribution function (cdf) of a rv Y is defined to be

$$F_Y(y) = P(Y \le y) \text{ for all } -\infty < y < \infty$$

Example: $Y \sim \text{Bernoulli}(p)$

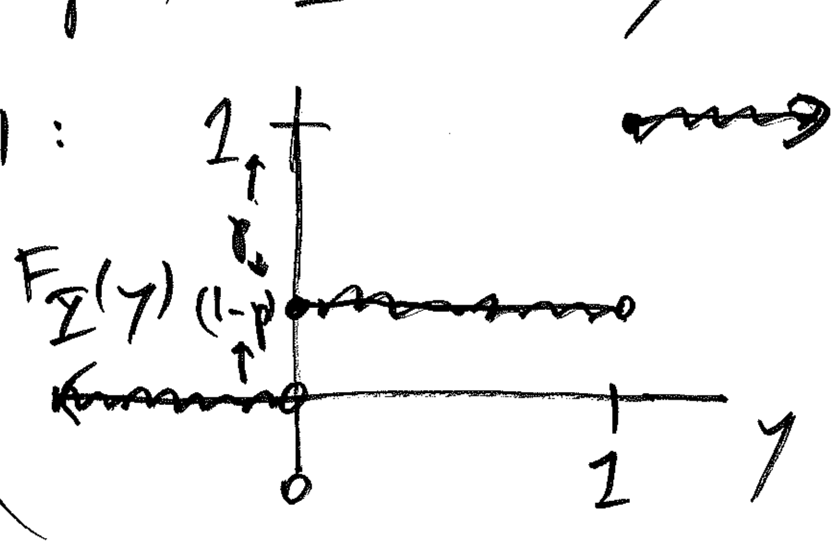
$$P(Y = y) = \begin{cases} p & \text{for } y = 1 \\ 1-p & 0 \\ 0 & \text{else} \end{cases}.$$

(PMF)

Notice that there's a clever way to write this pf:

$$P(Y = y) = p^{y}(1-p)^{1-y} I_{\{0,1\}}(y)$$

The cdf of Y is 0 for $y < 0$; at $Y = 0$ it jumps up to $(1-p)$ and stays there for $0 \le y < 1$; and at $Y = 1$ it jumps up to 1 & stays there for $y \ge 1$:



You can see that in general

$$0 \le F_Y(y) \le 1$$

and it's also clear that a cdf $F_Y(y)$ has to be a <u>non-decreasing function</u> of y: if $y_1 < y_2$ then $F_Y(y_1) \le F_Y(y_2)$.

Furthermore, $\lim_{y \to -\infty} F_Y(y) = 0$ and

$\lim_{y \to +\infty} F_Y(y) = 1$.

CDFs F_Y can be continuous on all of $\mathbb{R}$ (when Y is continuous) but certainly don't have to be (see the CDF of the Bernoulli (p) distribution).

Technical fact:

Def. $F_Y(y^-) \triangleq \lim_{\substack{y^* \to y \\ y^* < y}} F_Y(y^*) \triangleq \lim_{y^* \uparrow y} F_Y(y^*)$

limit from the left

(y^* goes to y from below)

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Def. $F_Y(y^+) \triangleq \lim_{y^* \to y,\ y^* > y} F_Y(y^*) \triangleq \lim_{y^* \downarrow y} F_Y(y^*)$

limit from right

(y^* going to y from above)

techical fact: $F_Y(y) = F_Y(y^+)$ for all $-\infty < y < \infty$

people call this <u>continuity from the right</u> or <u>continuity from above</u>

If $F_Y(y^-) = F_Y(y^+) = F_Y(y)$ then F_Y is continuous at y

Consequences of the CDF definition

① $P(Y > y) = 1 - F_Y(y)$

② For all y_1, y_2 with $y_1 < y_2$

$P(y_1 < Y \le y_2) = F_Y(y_2) - F_Y(y_1)$.

Consequence 2 means that if Y is continuous, there's an intimate connection between $F_Y(y)$ (cdf) and $f_Y(y)$ (pdf):

Y continuous: $y_1 < y_2$

$$P(y_1 < Y \le y_2) = \boxed{F_Y(y_2) - F_Y(y_1) = \int_{y_1}^{y_2} f_Y(y)\,dy}$$

and thus

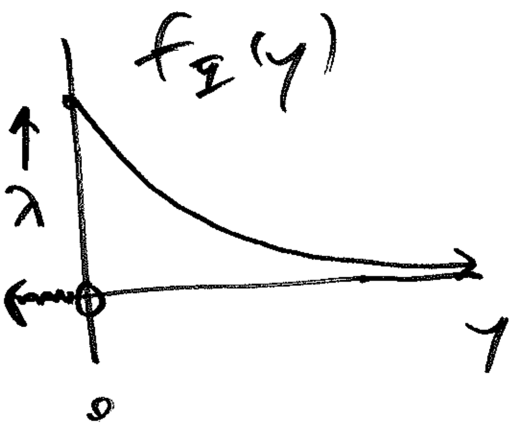
Theorem If Y is a continuous rv, with pdf $f_Y(y)$ and CDF $F_Y(y)$ then

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)\,dt$$ and $$\frac{d}{dy}F_Y(y) = f_Y(y).$$ at all continuity points of f

In other words Y continuous $\leftrightarrow$ the derivative of $\underset{\text{CDF}}{F_Y(y)}$ is $\overset{\text{pdf}}{f_Y(y)}$ (and $F_Y(y)$ is an anti-derivative (integral) of $f_Y(y)$.

Definition Y follows an <u>exponential distribution</u> $\mathcal{E}(\lambda)$ with (rate) parameter $\lambda > 0$

$$\leftrightarrow \underset{\text{pdf}}{f_Y(y)} = \begin{cases} \lambda e^{-\lambda y} & y > 0 \\ 0 & y \le 0 \end{cases}$$



The exponential dist. has a fundamental connection to the poisson distribution in poisson processes that we'll explore later.

It's easy to calculate the CDF of an exponential distribution:

Notation

$\underline{Y}$ exponentially distributed with parameter $\lambda > 0$ $\leftrightarrow$ $(\underline{Y} \mid \lambda) \to \underline{Y} \sim \text{Exponential}(\lambda)$

$\updownarrow$

$\underline{Y} \sim \mathcal{E}(\lambda)$

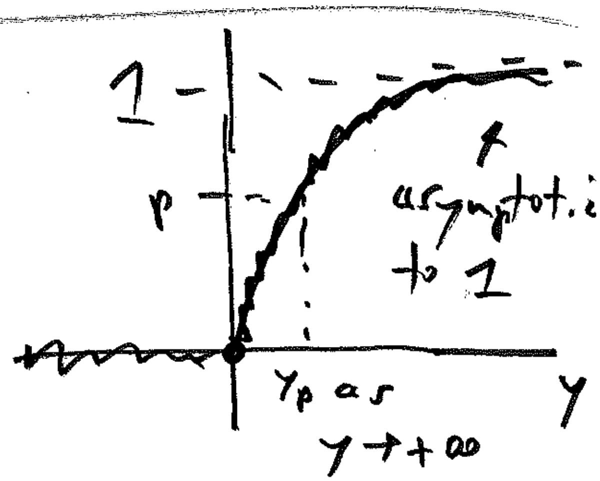
for $y > 0$

$$F_{\underline{Y}}(y) = \int_{-\infty}^{y} f_{\underline{Y}}(t)\,dt = \int_0^y \lambda e^{-\lambda t}\,dt$$

$$= \lambda \left. \frac{e^{-\lambda t}}{-\lambda} \right|_0^y = -1\left(e^{-\lambda y} - 1\right) = 1 - e^{-\lambda y}$$

So

$$F_{\underline{Y}}(y) = \begin{cases} 0 & \text{for } y \le 0 \\ 1 - e^{-\lambda y} & y > 0 \end{cases}$$



(30 Apr 19)