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AMS 131: Take-Home Test 1

Target due date: at `canvas.ucsc.edu` by 11.59pm Sun 28 Apr 2019 [*330 total points*]

Here's a style guide for all of the written work in this class. In figuring out how to write up answers to quizzes and take-home tests, pretend that the grader is sitting there with you and you're having a brief discussion with her/him on each question — that is, write down in a few sentences what you would say to someone to support your position. It's never enough in this class to just say “yes” or “10.3,” even if the right answer is “yes” or “10.3”; you need to say “yes (or 10.3), because” The right answer with no reasoning to support it, or the wrong reasoning, will get about half credit in this course, as will the wrong answer arrived at with a good effort. Leaving a problem or a part of a problem blank will get no credit.

This test is to be entirely your own efforts; do not collaborate with anyone or get help from anyone but me or our TAs (Laura Baracaldo, René Gutierrez, and Xingchen (Joe) Yu). The intent is that the course lecture notes and readings should be sufficient to provide you with all the guidance you need to solve the problems posed below, but you may use other written materials (e.g., the web, journal articles, and books other than those already mentioned in the readings), **provided that you cite your sources thoroughly and accurately**; you will lose (substantial) credit for, e.g., lifting blocks of text directly from `wikipedia` and inserting them into your paper without full attribution.

If it's clear that (for example) two people have worked together on a part of a problem that's worth 20 points, and each answer would have earned 16 points if it had not arisen from a collaboration, then each person will receive 8 of the 16 points collectively earned (for a total score of 8 out of 20), and I reserve the right to impose additional penalties at my discretion. If you solve a problem on your own and then share your solution with anyone else (because people from your cultural background routinely do this, or out of pity, or kindness, or whatever motive you may believe you have; it doesn't matter), you're just as guilty of illegal collaboration as the person who took your solution from you, and both of you will receive the same penalty. This sort of thing is necessary on behalf of the many people who do not cheat, to ensure that their scores are meaningfully earned. In an AMS graduate class I taught in 2012, five people failed the class because of illegal collaboration; don't let that happen to you.

1. [*55 points*] (public health) In 1972 a one-in-six random survey of the electoral roll — largely concerned with studying heart disease and smoking — was carried out in Whickham, a mixed urban and rural district near Newcastle upon Tyne in England. Twenty years later a follow-up study was conducted, with the results published in the journal *Clinical Endocrinology* in 1995.

The dataset summarized below in this problem pertains to the subsample of 1,314 women in the study who were classified in the original survey either as current smokers or as never having smoked. There were relatively few women (162) who had smoked but stopped, and only 18 whose smoking habits were not recorded; these women are not included in the data here. The 20-year survival status was determined for *all* the women in the original survey.

The *outcome* variable Y of interest here was mortality, recorded as dead or alive in 1992; the re-

searchers regarded X , smoking behavior in 1972 (current smoker or never smoked), as the *supposedly causal factor (SCF)*, and they also measured the variable Z , age (18–64 or 65+) in 1972.

Several definitions and conclusions from the field of *experimental design* are relevant here:

- A *controlled experiment* is a study in which the investigators have control over X , in the sense that they assign participants to different groups defined by X (in this case, smoker (the so-called *treatment group T*) versus never-smoked (the *control group C*)); controlled experiments become *randomized controlled trials (RCTs)* when the investigators assign the participants to T and C at random. Investigations in which the researchers have no control over who gets into T and C — typically because the participants themselves choose which group they’re in — are called *observational studies*.
- Two variables V and W are *associated* if as V increases W tends on average to increase or decrease, and vice versa; two variables that are not associated are *independent*. If both of the variables are *binary* — i.e., if they each have only two possible values, which may without loss of generality be taken as 0 and 1 — then $\{V \text{ and } W \text{ are associated}\} \iff \{\text{as } V \text{ moves from 0 to 1, } P(W = 1) \text{ increases or decreases}\}$.
- A *confounding factor (CF)* is a third variable Z , distinct from Y and X , that satisfies two properties:
 - Z and X are associated, and
 - Z and Y are associated.

The conclusion that changes in X *cause* changes in Y (at least probabilistically) may validly be drawn from RCTs, but not necessarily from observational studies, because of CFs: an apparent relationship between X and Y in an observational setting may in fact have been caused, in whole or in part, by a CF Z .

The best way to remove the possibility of a CF Z confounding your causal understanding is to *hold it constant*: to examine the relationship between X and Y separately for each possible value of Z — if you see something going on between Y and X in each of these comparisons, the association between X and Y cannot have been caused by Z , because it’s been held constant. This holding-constant process is called *controlling for the CF Z*.

Table 1: Age Group 18–64					Table 2: Age Group 65+			
Smoker?					Smoker?			
Mortality	Yes	No	Total		Mortality	Yes	No	Total
Dead	93	69	162		Dead	46	161	207
Alive	440	470	910		Alive	3	32	35
Total	533	539	1072		Total	49	193	242

Table 3: Overall			
Smoker?			
Mortality	Yes	No	Total
Dead	139	230	369
Alive	443	502	945
Total	582	732	1314

- (a) Is the investigation described in this problem a controlled experiment or an observational study? If it's a controlled experiment, is it an RCT? Explain briefly. [5 points]
- (b) Compute $P(\text{smoker})$ for a randomly chosen woman from Table 3, and compare this with your computation of $P(\text{smoker} | 18\text{--}64)$ for a woman picked at random from Table 1 and $P(\text{smoker} | 65+)$ for a women chosen at random from Table 2. Are age and smoking habits independent in this sample of 1,314 women, or does an association between these two variables exist in this data set (and if so, in which direction does the relationship go)? Explain briefly. [10 points]
- (c) For a woman chosen at random from the 1,314 in Table 3, compute $P(\text{dead})$, $P(\text{dead} | \text{smoker})$, and $P(\text{dead} | \text{nonsmoker})$. Does this establish an association between smoking and mortality for these women, and if so in which direction? Is the direction of this relationship surprising? Does this prove that smoking *causes* higher or lower mortality for these women? Explain briefly. [10 points]
- (d) By looking at Tables 1 and 2 and computing any relevant probabilities (unconditional or conditional), explain why age is a CF in studying the relationship between smoking and mortality for these 1,314 women. Separately for each of the age groups $\{18\text{--}64\}$ and $\{65+\}$ (i.e., for women chosen randomly from Tables 1 and 2), compute $P(\text{dead})$, $P(\text{dead} | \text{smoker})$, and $P(\text{dead} | \text{nonsmoker})$. How can you explain the fact that, when age is taken into consideration, the association between smoking and mortality for these women goes in the opposite direction than in part (b)? [15 points]
- (e) If the relationship between X and Y changes direction when a CF Z is controlled for, the situation is referred to as a *Simpson's Paradox* (named for the British statistician Edward Simpson (1922–2019), who wrote about it in 1951, although the phenomenon had been known about for a long time before that). By examining the directions of the relationships between (X, Y) , (X, Z) and (Y, Z) , explain intuitively why the Simpson's Paradox occurred here. Which conclusion about the effects of smoking on mortality is more trustworthy, the one in part (c) or its opposite in part (d)? Explain briefly. [15 points]

2. [70 points] (gambling) To solve this problem I need to tell you about *hypergeometric* probabilities (we'll revisit this topic in the unit on discrete distributions). Suppose that you're considering a finite population of individuals, each of which can be classified in one of two ways (e.g., black and green balls in an urn, or Democrats and Republicans among people who stick to the major political parties). Let the total number of individuals in the population be N , of which N_1 are of type 1 and N_2 of type 2 (with $N_1 + N_2 = N$). If you now take a simple random sample (without replacement) of size n from this population, what's the probability that you'll end up with exactly n_1 individuals of type 1 and n_2 of type 2?

Evidently there are some restrictions here: $0 \leq n_1 \leq N_1$, and $0 \leq n_2 \leq N_2$, and $n_1 + n_2 = n$. From our discussion of permutations and combinations, you can immediately see that there are $\binom{N}{n}$ possible simple random samples, all of which are equally likely, and furthermore that there are $\binom{N_1}{n_1}$ ways to choose the n_1 type-1 individuals and $\binom{N_2}{n_2}$ ways to end up with exactly n_2 individuals of type 2. Thus

Table 4: *The nine ways to win in Powerball and the associated “odds,” as stated on the Powerball website.*

Match	Prize	“Odds”
All five whites and the red	Grand Prize	1 in 292,201,338.00
All five whites	\$1,000,000	1 in 11,688,053.52
Four whites and the red	\$50,000	1 in 913,129.18
Four whites	\$100	1 in 36,525.17
Three whites and the red	\$100	1 in 14,494.11
Three whites	\$7	1 in 579.76
Two whites and the red	\$7	1 in 701.33
One white and the red	\$4	1 in 91.98
The red	\$4	1 in 38.32

$$P(n_1 \text{ type-1 individuals and } n_2 \text{ type-2 individuals}) = \frac{\binom{N_1}{n_1} \binom{N_2}{n_2}}{\binom{N}{n}}. \quad (1)$$

OK, now we can get on with the problem, which makes extensive use of these hypergeometric probabilities.

Powerball is a national lottery in the U.S. with drawings every Wednesday and Saturday night at 10.59pm Eastern time. The money left over after paying the winners is used by each state for projects designated by the legislatures, such as helping to fund K–12 education. In the *Powerball* game, five numbered white balls are drawn — in a manner certified by the lottery to be as close as humanly possible to *at random without replacement* — from a drum containing white balls numbered from 1 to 69, and one red ball is then also drawn at random from a second smaller drum that has 26 numbered red balls in it. Table 4 lists the nine ways you can win and the “odds” against you. Each play of the game cost \$2, and you can play as many times as you like.

There are several errors on the *Powerball* website. The first error is that when the *Powerball* people said “Odds” in Table 4 what they really meant was “the probability of occurrence, expressed as a fraction $\frac{1}{x}$.” Another error was present in something the *Powerball* website further stated:

The overall “odds” of winning a prize are 1 in 24.87. The “odds” presented here are based on a \$2 play (rounded to two decimal places) [*quotes added*].

- (a) Explain why the “odds” value in the first row of Table 4 was not $1 \text{ in } (69 \cdot 68 \cdot \dots \cdot 65 \cdot 26) = 35,064,160,560$, and why the stated “odds” value was essentially correct. [*10 points*]

(b) Explain why the “odds” value for the Second Prize of \$1,000,000 was not $\left(\frac{69}{5}\right)^{-1} = 1$ in 11,238,513, and show that the lottery people got the correct answer. [10 points]

(c) For $(k = 0, 1, \dots, 5)$, explain why the following formulas are correct:

$$\begin{aligned}
 P(k \text{ whites and the red}) &= \frac{\binom{5}{k} \binom{64}{5-k} \binom{1}{1} \binom{25}{0}}{\binom{69}{5} \binom{26}{1}} \text{ and} \\
 P(k \text{ whites (and not the red)}) &= \frac{\binom{5}{k} \binom{64}{5-k} \binom{1}{0} \binom{25}{1}}{\binom{69}{5} \binom{26}{1}}. \tag{2}
 \end{aligned}$$

Use these formulas to verify the rest of the “odds” entries in Table 4. [30 points]

(d) Show that the lottery people were right when they said that the overall “odds” of winning a prize are 1 in about 24.87, and explain why the statement “The “odds” presented here are based on a \$2 play (rounded to two decimal places)” initially sounds ridiculous but can be made correct with the insertion of a single word. [10 points]

(e) Suppose that T tickets were bought across the entire U.S. in a given week, that no one was clairvoyant or otherwise privy to knowledge about the winning numbers, and (for simplicity) that everybody made their lottery picks independently of everybody else. In the drawing on 30 Jul 2016, for which the Grand Prize (or *jackpot*) was \$487 million, it could be estimated from historical records on numbers of tickets purchased as a function of jackpot size that T was about 182.9 million. Show that the chance of at least one Grand Prize winner on this occasion was about 46.5%. (In actuality, one winning ticket was sold in a supermarket in Raymond, New Hampshire.) [10 points]

3. [30 points] (logic and Bayes’s Theorem) Here’s a small fictitious drama with five actors: three people — A , B and C — on death row; the governor, who has chosen one of them at random to be pardoned; and a warden in the prison, who knows the identity of the person the governor picked but isn’t allowed to tell A , B or C who the lucky person will be. Person A now speaks to the warden, as follows.

Please tell me the name of one of the other prisoners who’s *not* going to be pardoned — no harm done, since you won’t be identifying the lucky person. Let’s agree on these rules: if B will be pardoned, you say C ; if C will get the pardon, you say B ; and if I’m the lucky person, you toss a 50/50 coin to decide whether to say B or C .

The warden thinks it over and says “ B won’t get the pardon.” This is good news to A , because he secretly didn’t believe that the warden’s statement contains no information relevant to him: he thinks that, given what the warden said, his chance for the pardon has gone up from $\frac{1}{3}$ to $\frac{1}{2}$. Use Bayes’s Theorem to show that A ’s reasoning is incorrect, thereby working out whether there *was* information in what the warden said that’s relevant to A ’s probability of being pardoned. [25 points]

4. [80 points] (optimal hiring strategy) Here's an oversimplified version of a common problem for personnel managers that nevertheless contains elements of realism. You've advertised an open position in your organization, and $n \geq 1$ candidates have put their names forward for consideration. You want to hire the best candidate, but before interviewing any of them — suppose that their resumes don't provide strong information with which to create a ranking — each of them in your judgment has equal probability $\frac{1}{n}$ of being the best. It would be great if you could just interview all n of them, because you would then know for sure who's best, but (as with the tech sector, for example) this is a fast-moving hiring environment (by the time you get to the end and figure out that (say) candidate 3 is best, that person has probably already taken another job), so you need to be adaptive. Here are the ground rules:

- Once the interviews start, you can rank the candidates you've already seen, but you'll have no information about how the remaining candidates will fit into the ranking; and
- After each interview (because of the fast-moving environment), you either immediately hire the candidate you've just seen (and stop the interviewing process) or let that candidate go, with no opportunity to call her or him back.

Here's the adaptive strategy you've decided to use:

- To get information about the quality of the applicant pool, you pick a number $0 \leq r < n$, and you (callously) interview the first r candidates without intending to hire any of them.
- Beginning with the next candidate ($r + 1$), you continue interviewing until the current candidate is the best you've seen so far, at which point you stop the interviewing process and hire that candidate.
- If none of the candidates from $(r + 1)$ to n is best, you just throw up your hands and hire candidate n .

The goals in this problem are twofold: to compute the probability that you hire the best candidate with this strategy, and to choose r to maximize this probability. Let $A =$ (you hire the best candidate) and $B_i =$ (the best candidate is person i in the interviewing sequence).

- For any $i > r$, show that the probability that {the best candidate among the first i people interviewed is one of the first r people} is $\frac{r}{i}$. [10 points]
- Explain why $P(A | B_i) = 0$ for $i \leq r$, and (hard) show that $P(A | B_i) = \frac{r}{i-1}$ for $i > r$. (*Hint*: it helps to define the events $C_i =$ (you keep interviewing until you see candidate i)). [15 points]
- Having specified a value of r before interviewing begins, let $p_r = P(A)$ with the chosen r value, and show that
 - $p_0 = \frac{1}{n}$, and
 - for $0 < r < n$, $p_r = \frac{r}{n} \sum_{i=r+1}^n \frac{1}{i-1}$. (*Hint*: Use the results from part (b).)

[15 points]

- (d) On the way to finding the optimal value of r , define $q_r = (p_r - p_{r-1})$ for $r = 1, \dots, (n - 1)$ and show that q_r is a strictly decreasing function of r for $r > 0$. [15 points]
- (e) Use (d) to show that the value of r that maximizes p_r is the largest r such that $q_r > 0$. (Hint: For $r > 0$, from the definition of q_r , it helps to write $p_r = p_0 + \sum_{i=1}^r q_i$.) [10 points]
- (f) Use (e) to find the best value of r when $n = 10$ and the resulting optimal value of p_r . Does the adaptive hiring strategy examined in this problem look good to you? Explain briefly. [15 points]

(Remarkable fact (not part of what you're asked to show in this problem), for those of you who like to think about math: it turns out, weirdly, that for $0 < r < n$, $\sum_{i=r+1}^n \frac{1}{i-1} = \Psi(n) - \Psi(r)$, where $\Psi(x) \triangleq \frac{d}{dx} \ln \Gamma(x)$ is the *digamma* function.)

5. [95 points] (portfolio management) You're a portfolio manager at a hedge fund, meaning that you make investment decisions about other people's money. Naturally enough, the people whose money you're investing want to know how risky your investment decisions are. To this end, a standard metric in the investment industry is the *Value at Risk* (*VaR*) of a portfolio. Letting the continuous random variable X represent the (unknown) change in value of the portfolio in question over a fixed time horizon, for example one month, suppose that the PDF of X — in your judgment, based on the best current information — concentrates most of its probability on the positive part of the real number line \mathbb{R} ; in other words, in your judgment the portfolio will probably increase in value over the next month but may instead decrease. Let $Y = -X$, so that Y is the pessimistic side of the X coin (so to speak): if $X > 0$ with high probability then $Y < 0$ with the same high probability. To quantify the term “high,” let α be a small positive number, so that $(1 - \alpha)$ is close to 1; then the *VaR* of the portfolio is defined to be the $(1 - \alpha)$ quantile of the distribution of Y . The tough part of implementing this idea is pinning down the PDF of X ; in this problem you'll examine how sensitive the *VaR* is to this PDF specification. Let's take $\alpha = 0.01$ in what follows; this is a frequent choice in calculating *VaR* values.

Note that a portfolio based on sensible trading of stocks on the New York Stock exchange will typically appreciate at a rate of about 7% per year, which translates to a rate of about 0.6% per month; this implies that, if the portfolio is expected to increase in value by about \$10 million in the next month, which is consistent with the PDFs in part (b) below, the total value of the portfolio at the beginning of the month was about \$1.7 billion.

- (a) Let $F_X(x)$ and $F_Y(y)$ be the CDFs for the random variables X and Y , respectively. By definition, to say that $VaR = v$ means that $F_Y(v) = (1 - \alpha)$. Work out how F_Y depends on F_X , and use this to show that $VaR = -F_X^{-1}(\alpha)$. [10 points]
- (b) Suppose that in your judgment the support of X is $[-10, +20]$, where the units are in millions of dollars, and that you think that the PDF of X should be monotonically increasing on its support (in other words, if $20 \geq x_2 > x_1 \geq -10$ then in your view $P(X \doteq x_2) > P(X \doteq x_1)$, where \doteq means *is approximately equal to*). For each of the un-normalized PDFs below, compute the normalizing constant, make a rough (or refined) sketch of the PDF, compute the CDF, and work out the resulting *VaR*. Given the initial portfolio value of about \$1.7 billion, does *VaR* seem highly sensitive to you across this range of PDF shapes? Explain briefly. (Below, c is a generic constant [real number], not necessarily equal as you move from (i) to (ii) to (iii), and $f_X(x)$ is nonzero only on $[-10, +20]$.)

- (i) (triangular [linear]) $f_X(x)$ is linear with positive slope and passes through the points $(-10, 0)$ and $(20, c)$.
- (ii) (quadratic) $f_X(x)$ is quadratic and achieves its minimum at the point $(-10, 0)$.
- (iii) (exponential) $f_X(x) = c \exp\left(\frac{\lambda x}{10}\right)$, with $c = 0.005569078782$ and $\lambda = 1.717229651$ (chosen to make the PDF just slightly positive (0.001) at $x = -10$).

[45 points]

- (c) What if (i), (ii) and (iii) are all unrealistically cheerful about the fate of the portfolio over the next month? Repeat (b) with the two PDFs below. How would you describe the *VaR*'s sensitivity (e.g., moderately insensitive, or highly sensitive, or ...) across the entire range of the five PDFs you examined? Explain briefly.

- (i) (uniform [constant]) $X \text{ Uniform}(-10, 20)$.
- (ii) (triangular [linear]) $f_X(x)$ is linear with negative slope and passes through the points $(-10, c)$ and $(20, 0)$.

[40 points]